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ESTIMATES OF THE WEAK DISTANCE BETWEEN FINITE-DIMENSIONAL BANACH SPACES

BY

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ABSTRACT

We prove a variant of a theorem of N. Alon and V. D. Milman. Using it we construct for every n-dimensional Banach spaces X and Y a measure space Ω and two operator-valued functions $T\colon \Omega \to L(X,Y)$, $S\colon \Omega \to L(Y,X)$ so that $\int_\Omega S(\omega)\circ T(\omega)\,d\omega$ is the identity operator in X and $\int_\Omega \|S(\omega)\|\cdot\|T(\omega)\|\,d\omega = O(n^\alpha)$ for some absolute constant $\alpha < 1$.

We prove also that any subset of the unit n-cube which is convex, symmetric with respect to the origin and has a sufficiently large volume possesses a section of big dimension isomorphic to a k-cube.

Let X and Y be Banach spaces of the same dimension n. The Banach–Mazur distance between X and Y is defined as $d(X,Y) = \inf \|T\| \cdot \|T^{-1}\|$ over all invertible operators T from X to Y. Computing or estimating the distance between Banach spaces is one of the central problems in the Local theory. In 1984, N. Tomczak-Jaegermann introduced another distance between Banach spaces [T-J1]. First she defined a weak factorization norm of the identity operator of X through Y as

$$q(X,Y) = \inf \int_\Omega \|S(\omega)\| \cdot \|T(\omega)\| d\omega$$

where inf is taken over all measure spaces $(\Omega, d\omega)$ and all (measurable) maps $T\colon \Omega \to L(X,Y)$, $S\colon \Omega \to L(Y,X)$ so that $\int_\Omega S(\omega) \circ T(\omega) d\omega = \mathrm{id}_X$—the identity operator. The weak distance between X and Y is defined as $\mathrm{wd}(X,Y) = \max(q(X,Y), q(Y,X))$. It is obvious that for all Banach spaces X, Y of the

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same dimension $\mathrm{wd}(X,Y) \le d(X,Y)$. A rather simple observation is that the Banach–Mazur distance between the Euclidean space and any other space of the same dimension coincides with the weak distance [T-J1]. More facts about the weak distance between finite-dimensional Banach spaces and the connections with other branches of the Local theory are discussed in the review [L-M]. One of the most interesting questions concerning the notion of the Banach–Mazur distance was the estimation of the maximal distance between two Banach spaces of dimension n. A trivial consequence of a classical result of F. John is that this distance is not greater than n [T-J2]. In the famous paper [G] E. Gluskin proved the existence of an absolute constant c so that for all n there are two n-dimensional Banach spaces for which the distance between them is at least cn. For some probability space (Ω, Σ, μ) he defined a random function G from Ω to the set of convex symmetric bodies in $\mathbb{R}^n$ such that for every ω, $G(\omega)$ is the absolute convex hull of $3n$ vectors in $\mathbb{R}^n$. These vectors are supposed to be independently normally distributed with respect to the measure μ. He defined the space X_ω to be $\mathbb{R}^n$ with the unit ball $G(\omega)$ and showed that with probability close to 1, $d(X_\omega, X_{\omega'}) \ge cn, (\omega,\omega') \in \Omega\times\Omega$. N. Tomczak-Jaegermann proved that with a probability close to 1 the weak distance between two Gluskin spaces is less than $C\sqrt{n}$ for some absolute constant C. This fact as well as other considerations led to the following two questions:

(1) How much could the weak distance differ from the Banach–Mazur distance?

(2) What is the maximal weak distance between two spaces of dimension n?

As to the question (1) we note here that for every $\varepsilon > 0$ there exist two n-dimensional Banach spaces for which the weak distance between them is less than $1+\varepsilon$ and the Banach–Mazur distance is not less than $c(\varepsilon)n$ for some constant $c(\varepsilon)$. For two k-dimensional Banach spaces Y and Z define the spaces W_m, W'_m and W''_m as follows:

$$W_m = \underbrace{Y \oplus Y \oplus \ldots \oplus Y}_{m \text{ times}} \oplus \underbrace{Z \oplus Z \oplus \ldots \oplus Z}_{m \text{ times}}$$

$$W'_m = \underbrace{Y \oplus Y \oplus \ldots \oplus Y}_{(m+1) \text{ times}} \oplus \underbrace{Z \oplus Z \oplus \ldots \oplus Z}_{m \text{ times}}$$

$$W''_m = \underbrace{Y \oplus Y \oplus \ldots \oplus Y}_{m \text{ times}} \oplus \underbrace{Z \oplus Z \oplus \ldots \oplus Z}_{(m+1) \text{ times}}$$

Then the weak distance between the spaces W'_m and W''_m is at most $1+\frac{1}{m}$. Indeed, for $j = 1 \ldots m$, let P_j be a projection from W'_m onto W_m whose kernel is the j-s summand of W'_m and P_{m+1} be a projection on $\underbrace{Y \oplus Y \oplus \ldots \oplus Y}_{m \text{ times}}$ whose kernel is the sum of the last summand Y of W'_m and all the summands Z. It is obvious that the identity operator on the space W_m can be factorized through the space W''_m and the norm of this factorization will be 1. Combining the projections P_k, $k = 1, \ldots, (m+1)$, with this factorization one gets a weak factorization of the identity operator of W'_m through W''_m with the weak factorization norm $\frac{m+1}{m}$. From the other side if we take Gluskin spaces as the spaces Y and Z then it can be checked (using the technique of S. Szarek [S1]) that the Banach–Mazur distance between W'_m and W''_m will be at least $c(m)n$ with large probability.

The main part of this paper is devoted to question (2). Our result is the following

THEOREM 1: *The weak distance between two n-dimensional Banach spaces is not greater than $Cn^{\frac{13}{14}} \log^{\frac{15}{7}} n$ for some absolute constant C.*

In view of this theorem it is natural to conjecture that the weak distance is actually at most $C\sqrt{n}$ (with perhaps some added logarithmic factor).

The proof of the theorem is based on the combination of two ideas. The first due to J. Lindenstrauss and A. Szankowski [L-S1], [L-S2] is that of using block-Gaussian matrices. The second is a variant of the theorem of N. Alon and V. Milman [A-M]. This theorem gives a possibility to find subspaces well isomorphic to ℓ_∞^k in certain Banach spaces. The lemma we prove in this direction has also another consequence concerning spaces with large volume ratio. Let B_X be the unit ball of the n-dimensional Banach space X. We define the volume ratio of the space X as

$$\mathrm{vr}(X) = \min \left(\frac{\mathrm{vol}(B_X)}{\mathrm{vol}(\mathcal{E})} \right)^{1/n}$$

over all ellipsoids $\mathcal{E}$ contained in B_X. This parameter plays a crucial role in the Local Theory ([L-M], [P]). If $\mathrm{vr}(X) = 1$ then X is obviously Euclidean. If $\mathrm{vr}(X) \le C$ for some constant C then X has a subspace of dimension at least $\frac{n}{2}$, C'-isomorphic to Euclidean space with the constant C' depending only on C ([S2]). The maximal value of $\mathrm{vr}(X)$ over all n-dimensional spaces is of order $\sqrt{n}$. In [B] K. Ball proved that the only space with maximal volume ratio is ℓ_∞^n. Thus one can conjecture that if the volume ratio of the space X is proportional to the maximal volume ratio then X has a sufficiently large subspace well isomorphic to ℓ_∞^k. By considering the dual of the Gluskin space one can see that the dimension of such a space can not exceed $\sqrt{n}$. We prove here

THEOREM 2: *Let $a > 0$ and X be a Banach space of dimension n. If $\mathrm{vr}(X) \geq a\sqrt{n}$ then there exists a subspace Y of X of dimension at least $C(a)\frac{\sqrt{n}}{\log n}$, $C(a)\log n$-isomorphic to the space $\ell_\infty^{\dim Y}$.*

If K is a convex symmetric with respect to the origin subset of the unit cube B_∞^n and

$$\left(\frac{\mathrm{vol}(B_\infty^n)}{\mathrm{vol}(K)}\right)^{1/n} \leq a,$$

then $\mathrm{vr}(K) \geq c(a)\sqrt{n}$. So, we have

COROLLARY: *Let K be a convex symmetric with respect to the origin subset of B_∞^n and*

$$\left(\frac{\mathrm{vol}(B_\infty^n)}{\mathrm{vol}(K)}\right)^{1/n} \leq a.$$

Then K has a section of dimension at least $C(a)\frac{\sqrt{n}}{\log n}$, $C(a)\log n$-isomorphic to the cube.

Remark: Using an argument of James [J], one gets for every $\varepsilon > 0$ a section of dimension $n^{c(a,\varepsilon)/\log\log n}$ which is $(1+\varepsilon)$-isomorphic to the cube of its dimension. ∎

We shall use the standard notation (see [M-S], [L-M]). For a finite set J we denote by $|J|$ the cardinality of J. By g we denote a Gaussian vector in $\mathbb{R}^n$, the vector with independently standard normal distributed coordinates. B_2^n stands for the Euclidean ball of $\mathbb{R}^n$, B_X for the unit ball of the space X. The dimensions of all the spaces are integer numbers. If the dimension we get in some formula is not integer, the integer part should be taken. By $C, \bar{C}, c$ we denote absolute constants whose value may change from line to line. We start with the following

LEMMA: *Let X be a Banach space of dimension n, and let B_2^n be the ellipsoid of maximal volume contained in B_X. Let a, K be positive constants. Assume that there is an orthogonal projection $P\colon \ell_2^n \to \ell_2^n$ with rank $P \geq an$ so that $\mathbb{E}\|Pg\|_X \leq K$. Then there exists a subspace of X of dimension $m \geq \frac{\sqrt{n}}{C(a)K}$ which is $C(a)K$-isomorphic to the space ℓ_∞^m.*

This lemma is a variant of the main result of the paper [A-M] of N. Alon and V. Milman. The proof we give is however different. Instead of the combinatorial technique of [A-M] we use here estimates of a probabilistic nature such as the lemma of Slepian and the large submatrices principle of Bourgain and Tzafriri [B-T1].

The proof consists of three steps. First we construct an orthonormal system in ℓ_2^n with some special properties. Then we choose a random subsystem of this system so that the hypercube generated by the vectors of this subsystem is contained in the unit ball of X. Finally we use the large submatrices principle to derive the lemma itself.

Without loss of generality we may assume that X is a subspace of $L_p(\nu)$ for some $2 < p < \infty$ and ν. Indeed, one can slightly change the norm of X to embed it into the space $\ell_n^{2^n}$.

STEP 1: Let X, P, a and K be the same as in the lemma. Then there exist a probability space (Ω, Σ, μ), an embedding of X into the space $L_p(\mu)$ and m vectors $w_1, \ldots, w_m$, $m \geq \alpha(a)n$, so that

(i) $\alpha(a) \leq \|w_i\|_X \leq 1$ for $i = 1, \ldots, m$,

(ii) $\langle w_i, w_j \rangle_{\ell_2^n} = \delta_{ij}$—Kronecker's symbol for $i, j = 1, \ldots, m$,

(iii) $\left\| \left(\Sigma_{k=1}^m |w_k|^2 \right)^{1/2} \right\|_{L_\infty(\mu)} \leq C(a)K.$

The constants $\alpha(a), C(a)$ depend only on a.

The proof is similar to the proof of proposition 2.5 in [B-T2]. We give it for the sake of completeness.

Proof: By the theorem of Lewis [L] there exist a probability space (Ω, Σ, μ), an embedding of the space X into $L_p(\mu)$ and an orthonormal (in the sense of $L_2(\mu)$) system $\{z_k\}_{k=1}^n$ in X such that $\Sigma_{k=1}^n |z_k|^2 = n$. Clearly the same equality is satisfied for all complete orthonormal systems in X.

Let I be the identity operator from ℓ_2^n onto X. Then by the theorem of F. John [T-J2]

$$\tag{1} \begin{gathered} \|I\colon \ell_2^n \to X\| = 1 \\ \pi_2(I^{-1}\colon X \to \ell_2^n) = \sqrt{n} \end{gathered}$$

Let Y be the image of the projection P. We construct inductively a sequence of $n_1 = \frac{an}{3}$ vectors $w_1, \ldots, w_{n_1}$ in Y so that

(a) $\langle w_k, w_\ell \rangle_{\ell_2^n} = \delta_{k\ell}$ (We shall identify the vectors w_k with their images under the identity operator I^{-1}),

(b) $\langle w_k, w_\ell \rangle_{L_2(\mu)} = 0$ for $k \neq \ell$,

(c) $\|w_k\|_X = \|w_k\|_{L_p(\mu)} \geq \sqrt{\frac{a}{3}}$.

Suppose that the vectors $w_1, \dots, w_s, s < \frac{an}{3}$, are already defined and define the vector w_{s+1}. First define a subspace H of X as follows:

$$H = Y \cap [\operatorname{span}(w_1 \dots w_s)]^{\perp}_{\ell_2^n} \cap [\operatorname{span}(w_1 \dots w_s)]^{\perp}_{L_2(\mu)}.$$

Then $\dim H \geq \dim Y - 2s \geq an - 2s > \frac{an}{3}$, and so by (1)

$$\begin{aligned}
\sqrt{\frac{an}{3}} &\leq \pi_2(I \circ I^{-1}|_H \colon X \to X) \\
&\leq \|I|_{I^{-1}(H)} \colon \ell_2^n \to X\| \cdot \pi_2(I^{-1}|_H \colon X \to \ell_2^n) \\
&\leq \sqrt{n} \cdot \|I|_{I^{-1}(H)} \colon \ell_2^n \to X\|.
\end{aligned}$$

Thus one can find a vector $w_{s+1} \in H$ such that $\|w_{s+1}\|_{\ell_2^n} = 1$ and $\|w_{s+1}\|_X \geq \sqrt{\frac{a}{3}}$. Having constructed the vectors $w_1, \dots, w_{n_1}$ we renumerate them so that the sequence of the norms $\|w_k\|_{L_2(\mu)}$ becomes non-decreasing. Since the sequence is orthonormal in the space ℓ_2^n, one can decompose the Gaussian vector g into the sum

$$g = \sum_{k=1}^{n_1} g_k w_k + \bar{g}$$

where the g_k are independent normal variables and $\bar{g}$ is a Gaussian vector in the orthogonal complement of $\operatorname{span}\{w_k\}_{k=1}^{n_1}$ in the space ℓ_2^n. So

$$\begin{aligned}
K &\geq \mathbb{E}\|Pg\|_X \geq \mathbb{E}\| \sum_{k=1}^{n_1} g_k w_k \|_{L_p(\mu)} \geq \mathbb{E}\| \sum_{k=\frac{n_1}{2}}^{n_1} g_k w_k \|_{L_2(\mu)} \\
&\geq \mathbb{E}\Big(\sum_{k=\frac{n_1}{2}}^{n_1} |g_k|^2 \Big)^{1/2} \|w_{\frac{n_1}{2}}\|_{L_2(\mu)} \geq \sqrt{\frac{n_1}{2}} \cdot \mathbb{E}|g| \cdot \|w_{\frac{n_1}{2}}\|_{L_2(\mu)} = \sqrt{\frac{n_1}{\pi}} \|w_{\frac{n_1}{2}}\|_{L_2(\mu)}.
\end{aligned}$$

It follows from this estimate that for every $k \leq \frac{n_1}{2}$, $\|w_k\|_{L_2(\mu)} \leq \sqrt{\frac{\pi}{n_1}} K$. Hence

$$\sum_{k=1}^{n_1/2} |w_k|^2 \leq \frac{\pi K^2}{n_1} \cdot \sum_{k=1}^{n_1/2} \frac{|w_k|^2}{\|w_k\|^2_{L_2(\mu)}} \leq \frac{\pi K^2}{n_1} \cdot n = \frac{3\pi K^2}{a},$$

because the vectors $\frac{w_k}{\|w_k\|_{L_2(\mu)}}$ form an orthonormal system in the space $L_2(\mu)$.
∎

STEP 2: Let the space X and the sequence $\{w_k\}_{k=1}^m$ be as in the step 1. Then there exist a constant $\bar{C}(a)$ and a set $J \subset \{1, \ldots, m\}$ of at least $\sqrt{m}$ elements so that

$$\|\sum_{k \in J} a_k w_k\| \le \bar{C}(a) K \cdot \max_{k \in J} |a_k| \tag{2}$$

for every sequence $\{a_k\}_{k \in J}$.

Proof: Denote by $z_k(\omega), k = 1, \ldots, m$, the sequence of independent Bernoulli random variables taking the values 0 and 1 so that $P\{z_k = 1\} = \delta = \frac{1}{\sqrt{m}}$. Let $\frac{1}{p} + \frac{1}{q} = 1$.

Define $J = J(\omega)$ as the set of all indices k such that $z_k(\omega) = 1$. It will be proved that with positive probability $|J| \ge \sqrt{m}$ and (2) is satisfied. To do this consider the number

$$\sigma = \mathbb{E}_\omega \sup_{\|f\|_{L_q(\mu)}=1} \sum_{k=1}^m z_k(\omega) |\langle f, w_k \rangle|.$$

Then

$$\sigma \le \delta \cdot \sup_{\|f\|_{L_q(\mu)}=1} \sum_{k=1}^m |\langle f, w_k \rangle| + \mathbb{E}_\omega \sup_{\|f\|_{L_q(\mu)}=1} \sum_{k=1}^m (z_k(\omega) - \delta) |\langle f, w_k \rangle|.$$

Each summand will be estimated separately. By (iii),

$$\begin{aligned} \sup_{\|f\|_{L_q(\mu)}=1} \sum_{k=1}^m |\langle f, w_k \rangle| &\le \Big\| \sum_{k=1}^m |w_k| \Big\|_{L_p(\mu)} \\ &\le \sqrt{m} \Big\| \Big(\sum_{k=1}^m |w_k|^2 \Big)^{1/2} \Big\|_{L_\infty(\mu)} \le \sqrt{m} C(a) K. \end{aligned}$$

The second summand is the expectation of the supremum of the sum of the independent random vectors $(z_k(\omega) - \delta)|\langle f, w_k \rangle|$ in the space $\ell_\infty(B_{L_q(\mu)})$. The expectation of these vectors is 0, so for normal variables $g_k(\omega')$ independent of $z_k(\omega)$ we get

$$\begin{aligned} &\mathbb{E}_\omega \sup_{\|f\|_{L_q(\mu)}=1} \sum_{k=1}^m (z_k(\omega) - \delta) |\langle f, w_k \rangle| \\ &\qquad \le \sqrt{2\pi} \mathbb{E}_\omega \mathbb{E}_{\omega'} \sup_{\|f\|_{L_q(\mu)}=1} \sum_{k=1}^m (z_k(\omega) - \delta) g_k(\omega') |\langle f, w_k \rangle|. \end{aligned}$$

By the lemma of Slepian [P] it is less than

$$\begin{aligned}\sqrt{2\pi}\mathbb{E}_\omega\mathbb{E}_{\omega'} &\sup_{\|f\|_{L_q(\mu)}=1}\sum_{k=1}^m (z_k(\omega)-\delta)g_k(\omega')\langle f,w_k\rangle \\ &= \sqrt{2\pi}\mathbb{E}_\omega\mathbb{E}_{\omega'}\|\sum_{k=1}^m (z_k(\omega)-\delta)g_k(\omega')w_k\|_{L_p(\mu)} \\ &\le \sqrt{2\pi}\mathbb{E}_\omega\mathbb{E}_{\omega'}\|\sum_{k=1}^m z_k(\omega)g_k(\omega')w_k\|_{L_p(\mu)}+\sqrt{2\pi}\delta\mathbb{E}_{\omega'}\|\sum_{k=1}^m g_k(\omega')w_k\|_{L_p(\mu)}.\end{aligned}$$

Since the sequence $\{w_k\}_{k=1}^m$ is orthonormal in the space ℓ_2^n these expectations are not greater than K. So, one gets

$$\sigma \le K(\sqrt{2\pi}\cdot(1+\delta)+C(a)) = \bar{C}(a)K.$$

This guarantees the existence of ω so that the set $J=\{k\colon z_k(\omega)=1\}$ contains at least $\sqrt{m}$ elements and

$$\sup_{\|f\|_{L_q(\mu)}=1}\sum_{k\in J}|\langle f,w_k\rangle| \le 2\bar{C}(a)K.$$

Thus if $\{a_k\}_{k\in J}$ is an arbitrary sequence of real numbers then

$$\|\sum_{k\in J}a_kw_k\|_X = \sup_{\|f\|_{L_q(\mu)}=1}\sum_{k\in J}a_k\langle f,w_k\rangle \le 2\bar{C}(a)K\cdot\max_{k\in J}|a_k|$$

as claimed.

STEP 3: If the sequence $\{w_k\}_{k\in J}$, satisfies (2) and, for all $j\in J, \alpha(a)\le\|w_j\|\le 1$, then there exists a subset I of J so that $|I|\ge\frac{1}{\tilde{C}(a)K}|J|$ and the sequence $\{w_k\}_{k\in I}$ is $\tilde{C}(a)K$-equivalent to the standard basis of ℓ_∞^I.

Proof: For $i\in J$ let $w_i^*\in X^*$ be such that $\langle w_i^*,w_i\rangle=1$. The scalar products $\langle w_i^*,w_j\rangle$, $i,j\in J$, form the matrix $M=(m_{ij})_{i,j\in J}$ such that $m_{ii}=1$ and, by (2), $\sum_{i\in I}|m_{ij}|\le\frac{\bar{C}(a)K}{\alpha}$.

By [J-S] there exists a subset I of J so that $|I|\ge C(a)|J|$ and $\Sigma_{\substack{j\in I\\ j\ne i}}|m_{ij}|<\frac{\alpha}{2}$ for all $i\in I$. Let $\{a_i\}_{i\in I}$ be a sequence of real numbers and $|a_{i_0}|=\max_{i\in I}|a_i|$. Then

$$\Big\|\sum_{i\in I}a_iw_i\Big\|\ge|a_{i_0}|\cdot\alpha-\Big\|\sum_{\substack{i\in I\\ i\ne i_0}}a_iw_i\Big\|>|a_{i_0}|\alpha-\frac{\alpha}{2}\max_{\substack{i\in I\\ i\ne i_0}}|a_i|\ge\frac{\alpha}{2}\max_{i\in I}|a_i|. \qquad \blacksquare$$

We recall the definition of ℓ-ellipsoid. Let E be a Banach space. For a linear operator $u\colon \ell_2^n \to E$ denote by $\ell(u)$ the following norm:

$$\ell(u) = \Big(\mathbb{E}\|\sum_{k=1}^{n} g_k e_k\|^2\Big)^{1/2},$$

where g_k are independent Gaussian variables and $\{e_k\}_{k=1}^n$ is the standard basis of the space ℓ_n^2. For $v\colon E \to \ell_n^2$ define the dual norm to $\ell(\cdot)$:

$$\ell^*(v) = \sup\{\mathrm{tr}\,(v \circ u)|\ \ell(u) \le 1\}.$$

By the theorem of Lewis [P], for every invertible operator u, $\ell(u)\cdot\ell^*(u^{-1}) \ge n$ and there exists an operator u_0, such that $\ell(u_0) = \ell^*(u_0^{-1}) = \sqrt{n}$. The ellipsoid $u_0(B_n^2)$ is called the ℓ-ellipsoid of the space E. By a theorem of Pisier [P], for every operator $v\colon \ell_n^2 \to E$, $\ell(v^*) \le C\log n \cdot \ell^*(v)$ for some absolute constant C.

COROLLARY: *Let Y be an n-dimensional Banach space and assume that B_2^n is the ℓ-ellipsoid of Y. Then for every K at least one of the following is satisfied:*

(i) *there exists a subspace Y_0 of $\mathbb{R}^n$ of dimension $\frac{n}{3}$ so that*

$$\|\mathrm{id}|_{Y_0}\colon \ell_2^n \to Y\| \le \frac{\sqrt{n}}{K}, \quad \|\mathrm{id}|_{Y_0}\colon \ell_2^n \to Y^*\| \le \frac{\sqrt{n}\log n}{K},$$

(ii) *there exists a subspace Z of Y of dimension $\frac{\sqrt{n}}{CK}$, CK-isomorphic to $\ell_\infty^{\dim Z}$,*

(iii) *there exists a subspace $\tilde{Z}$ of Y^* of dimension $\frac{\sqrt{n}}{CK}$, CK-isomorphic to $\ell_\infty^{\dim \tilde{Z}}$.*

Proof: Let $\mathcal{E}$ be the ellipsoid of maximal volume contained in the unit ball of Y. Let $e_1,\dots,e_n$ be the axes of $\mathcal{E}$ and $\lambda_1,\dots,\lambda_n$, $\lambda_1 \le \lambda_2 \le \dots \le \lambda_n$, their lengths. Let T be the diagonal operator defined by $Te_i = \lambda_i e_i, 1 \le i \le n$, and thus $T(B_2^n) = \mathcal{E}$.

If $\lambda_{\frac{n}{3}} \le \frac{K}{\sqrt{n}}$ let $P\colon \mathbb{R}^n \to \mathbb{R}^n$ be the orthogonal projection onto the space $Y_1 = \mathrm{span}\{e_1,\dots,e_{\frac{n}{3}}\}$. Let g be the standard Gaussian vector in the space $\mathbb{R}^n$ and put $\tilde{g} = Tg$ the standard Gaussian vector in the space $(\mathbb{R}^n, \|\cdot\|_{\mathcal{E}})$. We have

$$\mathbb{E}\|P\tilde{g}\|_Y \le \|PT\|\cdot\mathbb{E}\|g\|_Y \le \frac{K}{\sqrt{n}}\cdot\ell(\mathrm{id}\colon \ell_2^n \to Y) = K$$

and thus the conditions of the lemma are satisfied. Hence by the lemma the case (ii) holds.

If $\lambda_{\frac{n}{3}} \geq \frac{K}{\sqrt{n}}$ then

$$\|\mathrm{id}|_{Y_1^\perp}: \ell_2^n \to Y\| \leq \|T^{-1}|_{Y_1^\perp}\| \leq \frac{\sqrt{n}}{K}.$$

Now let $\tilde{\mathcal{E}}$ be the ellipsoid of maximal volume contained in the unit ball of Y^*. As before let $\tilde{e}_1, \dots, \tilde{e}_n$ be the axes of $\tilde{\mathcal{E}}$ and $\tilde{\lambda}_1, \tilde{\lambda}_2, \dots, \tilde{\lambda}_n$, $\tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \dots \leq \tilde{\lambda}_n$ their lengths. By the same reasoning if $\tilde{\lambda}_{n/3} \leq \frac{K}{\sqrt{n}\log n}$ then case (iii) of the corollary holds. If also in this inequality the opposite is true then for the space $Y_2 = \mathrm{span}\{\tilde{e}_1, \dots, \tilde{e}_{\frac{n}{3}}\}$

$$\|\mathrm{id}|_{Y_2^\perp}: \ell_2^n \to Y^*\| \leq \frac{\sqrt{n}\log n}{K}.$$

Thus in this case (i) holds for the space $Y_0 = Y_1^\perp \cap Y_2^\perp$. ∎

Remark: One can prove the corollary in the following slightly stronger form: the case (i) should be changed to

(i)' *there exists a subspace Y_0 of $\mathbb{R}^n$ of dimension $\frac{n}{6}$ such that*

$$\|\mathrm{id}|_{Y_0}: \ell_2^n \to Y\| \leq \frac{\sqrt{n}}{K}, \quad \|\mathrm{id}|_{Y_0}: Y \to \ell_2^n\| \leq C,$$
$$\|\mathrm{id}|_{Y_0}: \ell_2^n \to Y^*\| \leq \frac{\sqrt{n}\log n}{K}, \quad \|\mathrm{id}|_{Y_0}: Y^* \to \ell_2^n\| \leq C.$$

Proof of Theorem 1: Let X and Y be two n-dimensional Banach spaces and assume that B_2^n is the ℓ-ellipsoid of X as well as of Y. Let K be a constant to be defined later. We have to estimate the weak factorization norm of the identity operator of X through Y.

We apply the corollary to the space Y. Suppose first that the case (i) of the corollary is satisfied. We define next a decomposition of $\mathbb{R}^n$ into a sum of orothogonal subspaces X_j, $j \leq 2\log_2 n$, so that for all j

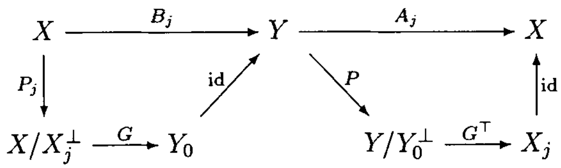
$$\|\mathrm{id}|_{X_j}: \ell_2^n \to X\| \cdot \|\mathrm{id}|_{X_j}: \ell_2^n \to X^*\| \leq 2\sqrt{n}. \tag{3}$$

Let $\mathcal{E}$ be the ellipsoid of maximal volume contained in the unit ball of X, $e_1 \dots e_n$ it's axes and $\lambda_1, \dots, \lambda_n$, $\lambda_1 \leq \dots \leq \lambda_n$, their lengths. We define the sequence m_j, and the subspaces X_j as follows:

$$m_1 = 1, \quad m_{j+1} = \min\{s > m_j \colon \lambda_s > 2\lambda_{m_j}\} \quad \text{if } \lambda_{m_j} < \frac{1}{2}\lambda_n$$

and $m_{j+1} = n+1$ if $\lambda_{m_j} \geq \frac{1}{2}\lambda_n, X_j = \mathrm{span}\{e_s : m_j \leq s < m_{j+1}\}$. We stop when $m_{j+1} = n+1$. The ellipsoids $X_j \cap \mathcal{E}$ are proportional to the unit ball, so the inequality (3) holds. It is obvious that there are at most $2\log_2 n$ spaces X_j.

Let Y_0 be a subspace of Y satisfying (i). Consider the following factorization:



Here P_j, P are the orthogonal projections onto the spaces X_j and $Y_0, G = (g_{ik})$, $1 \leq i \leq \frac{n}{3}$, $1 \leq k \leq \dim X_j$ a Gaussian matrix with independent normally distributed elements g_{ik}. One can easily check that

$$\mathbb{E}(A_j \circ B_j|_{X_j}) = \mathbb{E}(\mathrm{id} \circ G^T \circ P \circ \mathrm{id} \circ G \circ P_j|_{X_j}) = \mathbb{E}\Big(\sum_{i=1}^{n/3} g_{ik}^2\Big) \cdot \mathrm{id}|_{X_j} = \frac{n}{3}\mathrm{id}|_{X_j}.$$

Hence,

$$\mathrm{id}_X = \frac{3}{n}\sum_j \mathbb{E}(A_j \circ B_j).$$

Let us estimate the norm corresponding to this representation.

$$\begin{aligned}\mathbb{E}\,\|A_j\|\cdot\|B_j\| &\leq \mathbb{E}\|G\|\cdot\|G^T\| \leq (\mathbb{E}\,\|G\|^2)^{1/2}\cdot(\mathbb{E}\|G^T\|^2)^{1/2}\\ &\leq C\cdot\mathbb{E}\,\|G\|\cdot\mathbb{E}\,\|G^T\|.\end{aligned}$$

mince B_2^n is the ℓ-ellipsoid of the spaces X and Y,

$$\begin{aligned}\ell(\mathrm{id}|_{X_j}\colon \ell_2^n \to X^*) &\leq \ell(\mathrm{id}\colon \ell_2^n \to X^*) \leq C\sqrt{n}\log n,\\ \ell(\mathrm{id}|_{Y_0}\colon \ell_2^n \to Y) &\leq \ell(\mathrm{id}\colon \ell_2^n \to Y) = \sqrt{n}.\end{aligned}$$

By the inequality of Chevet [T-J2]

$$\begin{aligned}\mathbb{E}\|G\| \leq & \|\mathrm{id}|_{X_j}\colon \ell_2^n \to X^*\|\cdot\ell(\mathrm{id}|_{Y_0}\colon \ell_2^n \to Y)\\ &+ \|\mathrm{id}|_{Y_0}\colon \ell_2^n \to Y\|\cdot\ell(\mathrm{id}|_{X_j}\colon \ell_2^n \to X^*)\\ \leq & \sqrt{n}\cdot\|\mathrm{id}|_{X_j}\colon \ell_2^n \to X^*\| + C\sqrt{n}\log n\cdot\|\mathrm{id}|_{Y_0}\colon \ell_2^n \to Y_0\|\\ \leq & \sqrt{n}\cdot\|\mathrm{id}|_{X_j}\colon \ell_2^n \to X^*\| + C\frac{n\log n}{K}.\end{aligned}$$

Similarly,

$$\begin{aligned}\mathbb{E}\|G^T\| \le & \|\mathrm{id}|_{Y_0}\colon \ell_2^n \to Y^*\| \cdot \ell(\mathrm{id}|_{X_j}\colon \ell_2^n \to X) \\ & + \|\mathrm{id}|_{X_j}\colon \ell_2^n \to X\| \cdot \ell(\mathrm{id}|_{Y_0}\colon \ell_2^n \to Y) \\ \le & \frac{n\log n}{K} + C\sqrt{n}\log n \cdot \|\mathrm{id}|_{X_j}\colon \ell_2^n \to X\|.\end{aligned}$$

Finally, by (3) one gets

$$\begin{aligned}\mathbb{E}\|G\| \cdot \mathbb{E}\|G^T\| \le & \frac{Cn^2\log^2 n}{K^2} + \frac{Cn^{3/2}\log^2 n}{K} \cdot \|\mathrm{id}|_{X_j}\colon \ell_2^n \to X\| \\ & + \frac{n^{3/2}\log n}{K}\|\mathrm{id}|_{X_j}\colon \ell_2^n \to X^*\| + Cn^{3/2}\log n.\end{aligned}$$

Let now e be a unit vector in X_j such that $\|e\|_X = \sup\{\|\tilde{e}\|_X | \tilde{e} \in X_j, \|\tilde{e}\|_{\ell_2^n} = 1\}$ and P_e is the orthogonal projection onto $\mathrm{span}\{e\}$. Then

$$\begin{aligned}\|\mathrm{id}|_{X_j}\colon \ell_2^n \to X\| = \|\mathrm{id}\circ P_e\colon \ell_2^n \to X\| = \ell(\mathrm{id}\circ P_e\colon \ell_2^n \to X) \\ \le \ell(\mathrm{id}\colon \ell_2^n \to X) = \sqrt{n}\end{aligned}$$

and similarly

$$\|\mathrm{id}|_{X_j}\colon \ell_2^n \to X^*\| \le C\sqrt{n}\log n.$$

Thus, if the case (i) of the corollary holds for the space Y then

$$\mathbb{E}\|G\| \cdot \|G^T\| \le \frac{Cn^2\log^2 n}{K} + Cn^{3/2}\log n$$

and so

$$q(X,Y) \le \frac{Cn\log^3 n}{K} + Cn^{1/2}\log^2 n.$$

Suppose next that the case (ii) holds for Y. By [T-J1] there exists a decomposition of the identity operator of X $\mathrm{id}_X = \sum_{j=1}^N U_j$, so that, for every j, U_j can be factorized through a certain diagonal operator $\Delta_j\colon \ell_\infty^n \to \ell_2^n$:

$$\begin{array}{ccc} X & \xrightarrow{U_j} & X \\ {\scriptstyle A_j}\downarrow & & \uparrow{\scriptstyle B_j} \\ \ell_\infty^n & \xrightarrow{\Delta_j} & \ell_2^n \end{array}$$

Here $\|A_j\| \le 1$, $\|B_j\| \le 1$, $\Delta_j = \mathrm{diag}(\Delta_j^1 \dots \Delta_j^n)$ and

$$\sum_{j=1}^N \|\Delta_j\| \le C\sqrt{n}.$$

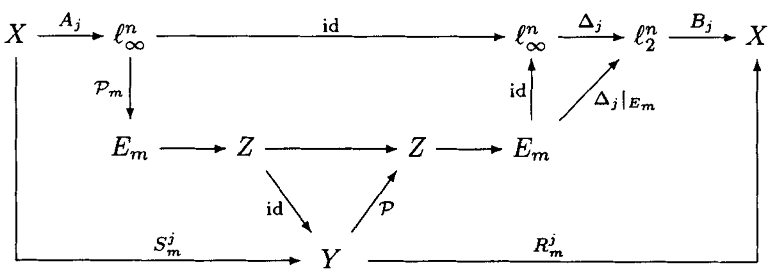
Decompose ℓ_∞^n into a direct sum of the spaces E_m, $\dim E_m = \frac{\sqrt{n}}{CK}$, $m = 1,\ldots,(CK\sqrt{n})$, spanned by elements of the standard basis of ℓ_∞^n. Let $\mathcal{P}_m$ be the natural projection of the space ℓ_∞^n onto E_m. Suppose that Z is the subspace of Y given by the case (ii) of the corollary. Then there exists a projection $\mathcal{P}$ onto Z such that

$$\|\mathcal{P}: Y \to Y\| \le CK.$$

Define a decomposition of the operator U_j as follows:

$$U_j = \sum_{m=1}^{CK\sqrt{n}} R_m^j \circ S_m^j$$

where the operators R_m^j and S_m^j are defined by the diagram



Then one gets $\|\Delta_j|_{E_m}\| = \left(\sum_{s=(m-1)\frac{\sqrt{n}}{CK}+1}^{m\frac{\sqrt{n}}{CK}} (\Delta_j^s)^2\right)^{1/2}$, so

$$\begin{aligned}\sum_{m=1}^{CK\sqrt{n}} \|R_m^j\| \cdot \|S_m^j\| &\le \sum_{m=1}^{CK\sqrt{n}} d(E_m, Z) \cdot \|\mathcal{P}\| \cdot \|\Delta_j|_{E_m}\| \\ &\le CK^2 \sum_{m=1}^{CK\sqrt{n}} \Bigg(\sum_{s=(m-1)\frac{\sqrt{n}}{CK}+1}^{m\frac{\sqrt{n}}{CK}} (\Delta_j^s)^2\Bigg)^{1/2} \\ &\le CK^2\left(CK\sqrt{n}\right)^{1/2}\left(\sum_{s=1}^{n}(\Delta_j^s)^2\right)^{1/2} = \tilde{C}K^{5/2}n^{1/4}\|\Delta_j\|.\end{aligned}$$

This means that weak factorization norm corresponding to the decomposition

$$\mathrm{id}_X = \sum_{j=1}^{N}\sum_{m=1}^{CK\sqrt{n}} R_m^j \circ S_m^j$$

is at most $CK^{5/2}n^{3/4}$.

If the case (iii) of the corollary holds then by duality one has the same estimate of the weak factorization norm. To end the proof choose the constant K in the optimal way, i.e. $K = Cn^{1/14}\log^{6/7} n$. This gives

$$q(X,Y) \le Cn^{13/14}\log^{15/7} n. \qquad \blacksquare$$

Remark: If $\dim X = n, \dim Y = m$ then the same calculations give the following estimate for the weak distance:

$$\mathrm{wd}(X,Y) \le C \max\left\{ \frac{n\log^{15/7} n}{m^{1/14}}; \frac{m\log^{15/7} m}{n^{1/14}} \right\}.$$

Proof of Theorem 2: Let B_2^n be the ellipsoid of maximal volume contained in the unit ball of X and $\mathcal{M}$ be the ℓ-ellipsoid of the space X^*. Let $e_1, \dots, e_n$ be the axes of $\mathcal{M}$ and $\mu_1, \dots, \mu_n, \mu_1 \le \dots \le \mu_n$, their lengths. We shall prove first that $\mu_{\frac{a^2}{9}n} \le \frac{3}{a\sqrt{n}}$. Assume that $\mu_{\frac{a^2}{9}n} > \frac{3}{a\sqrt{n}}$. Let Z be the subspace of X spanned by the vectors e_k, $1 \le k \le \frac{a^2}{9}n$, and P be the orthogonal projection onto the space Z. Denote by $X_{\mathcal{M}}$ the n-dimensional Banach space with the unit ball $\mathcal{M}$. Then we have

$$\|\mathrm{id}|_{Z^\perp}\colon \ell_2^n \to X_{\mathcal{M}}\| \le \frac{a\sqrt{n}}{3}. \tag{5}$$

By the lemma of Urysohn [P] one gets

$$\mathbb{E}\|g\|_{X^*} \ge \sqrt{n}\cdot\left(\frac{\mathrm{vol}(B_X)}{\mathrm{vol}(B_2^n)}\right)^{1/n} \ge an.$$

From the other side,

$$\mathbb{E}\|g\|_{X^*} \le \mathbb{E}\|Pg\|_{X^*} + \mathbb{E}\|(\mathrm{id}-P)g\|_{X^*}.$$

By (5),

$$\mathbb{E}\|(\mathrm{id}-P)g\|_{X^*} \le \frac{a\sqrt{n}}{3}\ell(\mathrm{id}|_{Z^\perp}\colon X_{\mathcal{M}} \to X^*) \le \frac{a\sqrt{n}}{3}\ell(\mathrm{id}\colon X_{\mathcal{M}} \to X^*) \le \frac{an}{3},$$

$$\mathbb{E}\|Pg\|_{X^*} = \sqrt{\dim Z}\int_{X^{n-1}\cap Z} \|z\|_{X^*}\,dm(z).$$

Here $m(z)$ is the Haar measure on the sphere $Z\cap S^{n-1}$. Since B_2^n is the ellipsoid of minimal volume which contains B_{X^*}, then, by the theorem of F. John, $\|z\|_{X^*} \le \sqrt{n}\|z\|_{\ell_2^n}$.

So,

$$\mathbb{E}\|Pg\|_{X^*} \le \sqrt{\dim Z} \cdot \sqrt{n} = \frac{a}{3} n$$

and we get the contradiction $\mathbb{E}\|g\|_{X^*} \le \frac{2}{3} an$. Thus we have proved that for the polar body $\mathcal{M}^0$ of the ellipsoid $\mathcal{M}$

$$\|\mathrm{id}|_Z\colon \ell_2^n \to X_{\mathcal{M}^0}\| = \mu_{\frac{a^2 n}{9}} \le \frac{3}{a\sqrt{n}}.$$

Since $\mathcal{M}$ is the ℓ- ellipsoid of X^*, we have

$$\ell(\mathrm{id}|_Z\colon X_{\mathcal{M}^0} \to X) \le \ell(\mathrm{id}\colon X_{\mathcal{M}^0} \to X) \le C\sqrt{n}\log n$$

and so

$$\|Pg\|_X \le \|\mathrm{id}|_Z\colon \ell_2^n \to X_{\mathcal{M}^0}\| \cdot \ell(\mathrm{id}|_Z\colon X_{\mathcal{M}^0} \to X) \le \frac{3c}{a}\log n.$$

Thus all the conditions of the lemma are satisfied. ∎

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